average slopes are illustrated in Figs. 2 and 3, with corresponding values of creep strain for the full 9000 hr. Undulations are noted which are believed to represent the material response to ambient temperature variations within the error of strain recording. Figure 4 is a composite arrangement of the welded and control specimen data which produces a straight line.

Concluding Discussion

Some of the limited test data available prior to this work were those obtained by Flanigan, Tedson, and Dorn.[1] Their work was conducted on R301-T (clad), which is a clad sheet-metal version of 2014-T6 alloy, at 94°F for times up to 1000 hr. Creep rates from their data of reference and from the present tests are plotted in Fig. 5. The two sets of data, taken together, provide a wide-range picture of the creep performance. The transition between the two portions would be an interesting subject for future investigation. The smallness of the creep strains experienced in these tests leads to the conclusion that creep is not a problem at the temperatures and stresses considered in the Titan II design.

Reference

[1] Flanigan, A. E., Tedson, L. F., and Dorn, J. E., "Stress rupture and creep tests on aluminum alloy sheet at elevated temperatures," American Institute of Mining, Metallurgical, and Petroleum Engineers Tech. Publ. 2033 (November 1946).

Radiation Pressure Torques from Spatial Variations in Surface Properties

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VARIOUS authors have treated the torque on spacecraft from solar radiation pressure, where the torque source arises from the asymmetry of the surface presented to the sun and differential shadowing effects (e.g., Ref. 1). Even when the surface is symmetrical, or at least balanced, with respect to the vehicle center of mass, a torque can exist because of differential reflectivity of surface elements and variations in the direction of the surface normal, both caused by local surface irregularities. These are statistical effects and may tend to average to zero over the entire vehicle. Nevertheless, there is a nonzero probability of torque magnitude different from zero. The purpose of this note is to develop the probability density function for torque components from this source.

With reference to Fig. 1, let zero be any base point used for the calculation of torque, e.g., the vehicle center of mass. Let $\{\mathbf{e}_\alpha\}$ ($\alpha = 1, 2, 3$) be a set of unit base vectors embedded in the vehicle with origin at O, and denote by S_i a surface element whose area is ΔA_i and whose unit normal vector is $\mathbf{n}_i$. S_i is located with respect to O by vector $\mathbf{r}_i$. Discrete, not infinitesimal areas, are considered here; the physical model corresponding to this assumption is an array of solar cells whose properties might reasonably be assumed constant but statistically different from those of neighboring cells. The argument is generalized in a straightforward way to a surface with continuous properties, in which case ΔA_i, in the analysis below, can be interpreted simply as the characteristic sizes of surface areas over which reflectivity and surface normal are substantially constant. Denote by $\mathbf{E}$ the unit vector in the direction of incident radiation, the same for all surface elements, and by $\mathbf{R}_i$ the unit vector in the direction of the net reflected momentum. It is supposed that $\mathbf{E}$, $\mathbf{n}_i$, and $\mathbf{R}_i$ are coplanar with angles $\langle \mathbf{E}, \mathbf{n}_i \rangle = \pi - \alpha_i$, $\langle \mathbf{n}_i, \mathbf{R}_i \rangle = \alpha_i'$. The angles α_i and α_i' need not be equal, if one assumes that the area ΔA_i itself has minute variations in surface normal that cause diffuse reflection. However, it is assumed here that, although the various $\mathbf{n}_i$ may differ among themselves, the reflection from any one area element is specular so that $\alpha_i' = \alpha_i$.

Take the incident radiation pressure to be P_0 (units of dynes/square centimeter, say) and the effective pressure of the reflected radiation momentum from element i to be $\eta_i P_0$, where η_i is a coefficient $0 \leq \eta_i \leq 1$ describing reflectivities from zero (complete absorption) to complete reflection. Noting that the projections of ΔA_i normal to $\mathbf{E}$ and $\mathbf{R}_i$ are $\Delta A_i \cos\alpha_i$ and $\Delta A_i \cos\alpha_i'$, respectively, it is evident that the torque about O of forces on ΔA_i is

$$\mathbf{L}_i = \mathbf{r}_i \times [P_0 \Delta A_i \cos\alpha_i \mathbf{E} + \eta_i P_0 \Delta A_i \cos\alpha_i' (-\mathbf{R}_i)] \quad (1)$$

Since $\mathbf{R}_i$, $\mathbf{E}$, and $\mathbf{n}_i$ are coplanar,

$$\mathbf{R}_i = \mathbf{E} + 2\cos\alpha_i \, \mathbf{n}_i \quad (2)$$

where the equality of α_i and α_i' has been used. Hence,

$$\mathbf{L} = P_0 \sum_i \Delta A_i \mathbf{r}_i \times [(1 - \eta_i)\cos\alpha_i \mathbf{E} - 2\eta_i \cos^2\alpha_i \mathbf{n}_i] \quad (3)$$

The scalar component of torque is found by introducing†

$$\mathbf{L} = L_\beta \mathbf{e}_\beta \qquad \mathbf{r}_i = r_{i\beta}\mathbf{e}_\beta \qquad \mathbf{E} = E_\beta \mathbf{e}_\beta \qquad \mathbf{n}_i = n_{i\beta}\mathbf{e}_\beta$$

There is no loss in generality in supposing $\mathbf{e}_3$ coincident with $-\mathbf{E}$, so that $E_\beta = -\delta_{3\beta}$. Then,

$$L_\beta = -P_0 \epsilon_{\beta\lambda\mu} \sum_i \Delta A_i r_{i\lambda} [(1 - \eta_i)\cos\alpha_i \delta_{3\mu} + 2\eta_i \cos^2\alpha_i n_{i\mu}] \quad (4)$$

Effects of Surface Variations

Suppose that η_i^*, α_i^*, and $\mathbf{n}_i^*$ are nominal values of η_i, α_i, and $\mathbf{n}_i$ such that, if these nominal values hold, the torque components $L_\beta = 0$. Here, we are concerned only with such *nominally balanced systems*: nominal unbalance can be treated separately by conventional methods. Suppose further that *actual* values of η_i, α_i, and $\mathbf{n}_i$ are

$$\eta_i = \eta_i^* + \delta\eta_i \qquad \alpha_i = \alpha_i^* + \delta\alpha_i$$
$$\mathbf{n}_i = \mathbf{n}_i^* + \delta\mathbf{n}_i$$

Note that $\delta\mathbf{n}_i$ is not arbitrary, since both $\mathbf{n}_i$ and $\mathbf{n}_i^*$ are unit vectors: thus $2\delta\mathbf{n}_i \cdot \mathbf{n}_i^* + |\delta\mathbf{n}_i|^2 = 0$. Nor is it independent

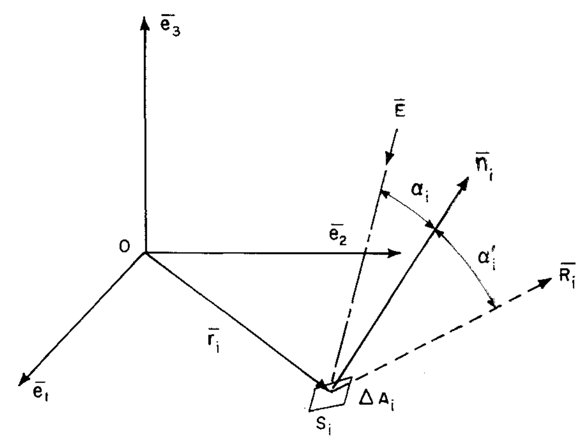

Fig. 1 Correlation of transition to turbulent flow in hypersonic wakes behind blunt bodies.

† The summation convention is used for Greek indices that have the range 1, 2, 3; Latin indices range over all of the surface elements, and the summation sign is included explicitly.

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of $\delta\alpha_i$, since $-\mathbf{E}\cdot\mathbf{n} = \cos(\alpha_i^* + \delta\alpha_i)$ whilst $-\mathbf{E}\cdot\mathbf{n}^* = \cos\alpha_i^*$. These restrictions can be written as two scalar constraint equations (neglecting $|\delta\mathbf{n}_i|$ with respect to unity) as

$$\eta_{i\rho}^*\delta n_{i\rho} = 0 \qquad (5)$$

$$E_\rho \delta n_{i\rho} = \sin\alpha_i^*\delta\alpha_i \qquad (6)$$

(Neither of these last equations is summed on i).

The variation in $L\beta$ using Eqs. (4) and (6) is

$$\delta L_\beta = P_0\epsilon_{\beta\lambda\mu}\sum_i \Delta A_i r_{i\lambda}\{[(1-\eta_i^*)\sin\alpha_i^*\delta_{3\mu} + 2\eta_i^*\sin 2\alpha_i^* n_{i\mu}^*]\delta\alpha_i - 2\eta_i^*\cos^2\alpha_i^*\delta n_{i\mu} + [\cos\alpha_i^*\delta_{3\mu} - 2\cos^2\alpha_i^* n_{i\mu}^*]\delta\eta_i\} = -P_0\epsilon_{\beta\lambda\mu}\sum_i \Delta A_i r_{i\lambda}(N_{i\mu\rho}\delta n_{i\rho} + H_{i\mu}\delta\eta_i) \qquad (7)$$

where

$$N_{i\mu\rho} = 2\eta_i^*\cos^2\alpha_i^*\delta_{\mu\rho} + [(1-\eta_i^*)\delta_{3\mu} + 4\eta_i^*\cos\alpha_i^* n_{i\mu}^*]\delta_{3\rho} \qquad (8)$$

$$H_{i\mu} = -\cos\alpha_i^*\delta_{3\mu} + 2\cos^2\alpha_i^* n_{i\mu}^* \qquad (9)$$

The variation $\delta\alpha_i$ has been eliminated in deriving Eq. (7), but the $\delta n_{1\rho}$ are still subject to the constraint of Eq. (5).

Expected Value of Torque

If $\delta\eta_i$ is a random variable with zero mean, the expected value‡ $\overline{\delta L_\beta}$ of δL_β is

$$\overline{\delta L_\beta} = -P_0\epsilon_{\beta\lambda\mu}\sum_i \Delta A_i r_{i\lambda}N_{i\mu\rho}\overline{\delta n_{i\rho}} \qquad (10)$$

In general, it is not valid to assume all $\overline{\delta n_{i\rho}} = 0$, although this satisfies the necessary condition, Eq. (5). However, in certain important special cases, one can reduce Eq. (10) further. Suppose, for example, that $\mathbf{n}_i$ is equally likely to lie anywhere in a small cone about $\mathbf{n}_1^*$, whence the expected values of the components of $\delta\mathbf{n}_i$ in the plane normal to $\mathbf{n}_i^*$ are zero. But Eq. (5) requires that any small variation $\delta\mathbf{n}_i$ shall lie precisely in that plane, i.e., that it shall have no component along $\mathbf{n}_i^*$. Thus, the vector equation $\overline{\delta\mathbf{n}_i} = 0$ holds, the components of $\delta\mathbf{n}_i$ with respect to $\mathbf{e}_\mu$ also have zero expected value, and it follows that $\overline{\delta L_\beta} = 0$.

Stochastic Model and Correlation Functions

Assume that the random variable $\delta\eta_i$ is uncorrelated with random variables $\delta n_{j\alpha}$ (where j may be the same as i, or not) so that $\overline{\delta\eta_i\delta\eta_{j\alpha}} = 0$ for all i, j, and α, and that the reflectivity variations in different surface elements are uncorrelated§ so that $\overline{\delta\eta_i\delta\eta_j} = \sigma_\eta^2\delta_{ij}$ (σ_η = the rms variation in reflectivity). Furthermore, suppose that variations in the normal are uncorrelated from surface to surface element¶ so that $\overline{\delta n_{i\rho}\delta n_{j\theta}} = \phi_{i\rho\theta}{}^N\delta_{ij}$ (not summed on i), where $\phi_{i\rho\theta}{}^N$ is the correlation function between the ρth and θth components of $\delta\mathbf{n}_i$. None of these assumptions as to lack of correlation is essential; they can be relaxed very easily with no penalty except increased algebra.

Under these conditions, Eq. (7) can be used to derive the torque component correlation function $\phi_{\beta\gamma}{}^L = \overline{\delta L_\beta\delta L_\gamma}$. The result is

$$\phi_{\beta\gamma}{}^L = P_0{}^2\epsilon_{\beta\lambda\mu}\epsilon_{\gamma\sigma\tau}\sum_i (\Delta A_i)^2 r_{i\lambda}r_{i\sigma}[N_{i\mu\rho}N_{i\tau\sigma}\phi_{i\rho\theta}{}^N + H_{i\mu}H_{i\tau}\sigma_{Hi}{}^2] \qquad (11)$$

where

$$\phi_{i\rho\theta}{}^N = \overline{\delta n_{i\rho}\delta n_{i\theta}} \quad \text{and} \quad \sigma_{Hi}{}^2 = \overline{(\delta\eta_i)^2}$$

Note that δL_β is the sum of a large number of random variables, at least if the number of elementary areas is not too small, so that, by the central limit theorem,** we are assured that the probability distribution of δL_β is quasi-normal. Thus it is completely characterized by the mean and correlation functions given by Eqs. (10) and (11), respectively.

A Special Case

A simple but important special case is that of normal incidence on a plane area, for which $N_{i\mu\rho} = \{2\eta_i^*\delta_{\mu\rho}(\mu = 1, 2);\ (1 + 5\eta_i^*)\delta_{\mu\rho}(\mu = 3)\}$ and $H_{i\mu} = \delta_{3\mu}$. Also, from Eq. (5), $\delta n_{i3} = 0$. From Eq. (11),

$$\phi_{\beta\gamma}{}^L = P_0{}^2\epsilon_{\beta\lambda\mu}\epsilon_{\gamma\sigma\tau}\sum_i (\Delta A_i)^2 r_{i\lambda}r_{i\sigma}\{(2\eta_i^*)^2[\delta_{\mu1}\delta_{\tau1}\phi_{i11}{}^N + \delta_{\mu1}\delta_{\tau2}\phi_{i12}{}^N + \delta_{\mu2}\delta_{\tau1}\phi_{i21}{}^N + \delta_{\mu2}\delta_{\tau2}\phi_{i22}{}^N] + \delta_{3\mu}\delta_{3\tau}\sigma_{Hi}{}^2\} \qquad (12)$$

Note especially the following subcases.

Case a: Complete absorption, $\eta_i^* = 0$,

$$\phi_{\beta\gamma}{}^L = P_0{}^2\epsilon_{\beta\lambda3}\epsilon_{\gamma\sigma3}\sum_i (\Delta A_i)^2 r_{i\lambda}r_{i\sigma}\sigma_{Hi}{}^2 \qquad (13a)$$

that is, the torque correlation function is independent of the tilt in the normal.

Case b: Nominal and stochastic properties independent of location on surface, $\eta_i^* = \eta^*$, $\phi_{i\alpha\beta}{}^N = \phi_{\alpha\beta}{}^N$, $\sigma_{Hi} = \sigma_H$,

$$\phi_{\beta\gamma}{}^L = (P_0A)^2\epsilon_{\beta\lambda\mu}\epsilon_{\gamma\sigma\tau}\{4\eta^{*2}[\delta_{\mu1}\delta_{\tau1}\phi_{11}{}^N + \delta_{\mu1}\delta_{\tau2}\phi_{12}{}^N + \delta_{\mu2}\delta_{\tau1}\phi_{21}{}^N + \delta_{\mu2}\delta_{\tau2}\phi_{22}{}^N] + \delta_{3\mu}\delta_{3\tau}\sigma_H{}^2\}k_{\lambda\sigma}{}^2 \qquad (13b)$$

where A is the entire surface area, and

$$k_{\lambda\sigma}{}^2 = \sum_i \frac{(\Delta A_i)^2 r_{i\lambda}r_{i\sigma}}{A^2} \qquad (14)$$

Note that the product (P_0A) is simply the total *incident* radiation force on the array. The $k_{\lambda\sigma}$ is rather analogous to a radius of gyration tensor, except that it measures a second moment of squared area rather than of mass. If the origin of the $\{\mathbf{e}_\alpha\}$ frame is in the plane surface, r_{i3} and all $k_{\alpha\beta}$ with α or β equal to three, must vanish.

If Eq. (13b) is expanded, one finds

$$\phi_{11}{}^L = k_{22}{}^2\sigma_H{}^2(P_0A)^2 \qquad (15a)$$

$$\phi_{22}{}^L = k_{11}{}^2\sigma_H{}^2(P_0A)^2 \qquad (15b)$$

$$\phi_{33}{}^L = 4\eta^{*2}(k_{22}{}^2\phi_{11}{}^N - 2k_{12}{}^2\phi_{12}{}^N + k_{11}{}^2\phi_{22}{}^N)(P_0A)^2 \qquad (15c)$$

$$\phi_{12}{}^L = -k_{12}{}^2\sigma_H{}^2(P_0A)^2 \qquad (15d)$$

$$\phi_{23}{}^L = \phi_{31}{}^L = 0 \qquad (15e)$$

Since the orientation of base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ has not yet been specified, there is no loss in generality in disposing them so that $\phi_{12}{}^L = 0$ as well.

As a specific numerical example for case b, suppose $\eta^* = 1$, $\sigma_H = 5\%$, $\phi_{11}{}^N = \phi_{22}{}^N = (0.5°)^2$ with $\phi_{12}{}^N = 0$. Also, let $P_0 = 0.3$ dynes/m², $\Delta A_i = 10$ cm², and suppose that the array is a rectangle with dimensions 6×10 m, point O at the center, long axis of the rectangle in direction $\mathbf{e}_2$. Then, $k_{11}{}^2 = 0.5$ cm², $k_{22}{}^2 = (25/18)$ cm², $P_0A = 18$ dynes, $\phi_{11}{}^N = \phi_{22}{}^N = 76 \times 10^{-6}$. From Eqs. (15) it follows that rms δL_α is of the order of 1 dyne-cm for $\alpha = 1, 2, 3$. Torques of this size are of no practical significance to such an array, so that one may conjecture for this numerical example that torques from surface variations are far below those from gross asymmetries or shadowing of the surface. However, other cases must be investigated before it can be concluded that this is a correct general statement. In particular, a large

‡ Over-bars denote expected values over the probability distribution for the variable in question.

§ This may not be strictly valid if a reflectivity degradation is produced by an effect correlated over a large surface area. However, in this case one would expect $\delta\eta_i$ not to be zero and these means to be correlated over many surface elements, but the variations about the mean degradation not necessarily to be correlated.

¶ This precludes shifts in the nominal normals caused by structural warpage, for example.

** See Ref. 2, for example.

change in the size of elementary areas over which the variations are correlated can have a significant effect. For example, if ΔA in this numerical example were 1 m^2 rather than 10 cm^2, the rms torque components would increase by a factor of 100, bringing them up to a more interesting level.

References

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[2] Cramer, H., *Mathematical Methods of Statistics* (Princeton University Press, Princeton, N. J., 1946), p. 213.

Boost-Phase Equilibrium Pressures in a Dual-Thrust Solid-Propellant Rocket Motor

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Nomenclature

a	= (sustainer) burning-rate coefficient
A	= area, in.^{2}
b	= (booster) burning-rate coefficient
C_D	= mass flow coefficient, lbm/lbf-sec
k	= specific-heat ratio
m	= (booster) pressure exponent
$\dot{m}$	= mass flow rate, lbm/sec
M	= Mach number
n	= (sustainer) pressure exponent
P	= pressure, psia
r	= burning rate, in./sec
R	= gas constant, lbf-ft/lbm-°F
s	= propellant surface area, in.^{2}
t	= time, sec
T	= temperature, °R
v	= volume, in.^{3}
w	= propellant web thickness, in.
ρ	= density (of solid propellant), lbm/in.^{3}

Subscripts

b	= booster
c	= stagnation conditions
e	= (nozzle) exit section
s	= sustainer
si	= subsonic (isentropic) sustainer nozzle
ss	= sonic sustainer nozzle
$(\ )^*$	= (nozzle) throat section

AFTER boost to a desired velocity, a sustain capability equal to drag allows a vehicle to fly a "vacuum" trajectory, thereby increasing both range and accuracy. Providing a sustain capability requires some form of dual-thrust propulsion system; the use of a dual-nozzle configuration (Fig. 1) offers one solution. Simultaneous boost-sustain ignition allows maximum reliability by eliminating a dual-ignition system. However, qualitative consideration of the combustion process indicates that the boost chamber pressure could impede the sustainer flow, causing some sustainer overpressurization. In turn, the sustainer gases entering the boost chamber cause the boost pressure to rise above design values. This note examines the equilibrium pressures attained in each motor during the boost phase and defines the influencing parameters.

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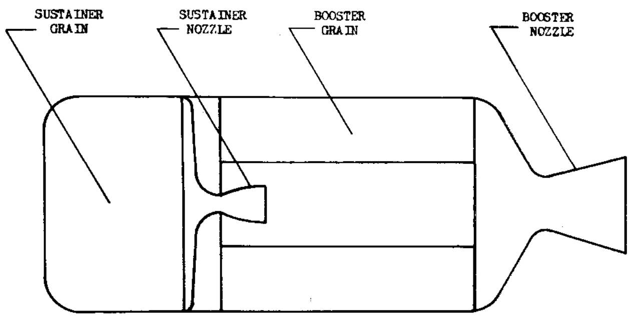

Fig. 1 Dual-nozzle rocket motor.

Design Parameters

Sustain thrust is 4% of boost, with both motors designed for 1000 psia P_c. Thus sustainer operation is influenced largely by the flow characteristics of a converging-diverging nozzle discharging into a region of variable back pressure.[1] If $P_{cs} \lesssim 1.1\ P_{cb}$, the nozzle acts as a subsonic venturi. For higher P_{cs}, $M = 1.0$ at the nozzle throat, the flow rate is constant (independent of P_{cb}), and P_{cb} is achieved through a normal shock in the divergent section.

Transient State

Assuming one-dimensional isentropic flow of a perfect gas through the combustion chamber, the conservation of mass yields the usual[2] solid-propellant design equation:

$$\frac{P_c}{RT} rS + \frac{v}{RT^2}\left(T\frac{dP_c}{dt} - P_c\frac{dT}{dt}\right) = \dot{m}_{\text{in}} - \dot{m}_{\text{out}} = \rho rS - C_D P_c A* \quad (1)$$

Assuming negligible variations in combustion temperature, and burning rate given by $r = aP_c{}^n$, Eq. (1) becomes

$$\frac{dP_c}{dt} = \frac{RT}{V}\left[\rho a P_c{}^n S\left(1 - \frac{P_c}{\rho RT}\right) - C_D P_c A*\right] \quad (2)$$

where $V = V_0 + \int aP_c{}^n S dt$.

From (2), the rate of change of boost and sustain chamber pressures are given by

$$\frac{dP_b}{dt} = \frac{R_b T_b}{V_b}\left[bP_b{}^m S_b\left(\rho_b - \frac{P_b}{R_b T_b}\right) + \dot{m}_s - C_{Db}P_b A_{b*}\right] \quad (3)$$

$$\frac{dP_s}{dt} = \frac{R_s T_s}{V_s}\left[aP_s S_s\left(\rho_s - \frac{P_s}{R_s T_s}\right) - \dot{m}_s\right] \quad (4)$$

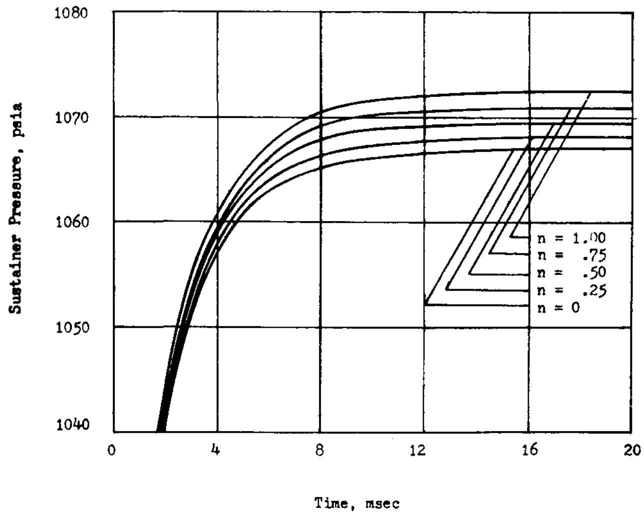

Fig. 2 Transient-state sustainer pressures for various values of pressure exponent (n).